

The quantum free particle on spherical and hyperbolic spaces: A curvature dependent approach.

José F. Cariñena^{† a)}, Manuel F. Rañada^{† b)}, Mariano Santander^{‡ c)}

[†] *Departamento de Física Teórica and IUMA, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, Spain*

[‡] *Departamento de Física Teórica, Facultad de Ciencias
Universidad de Valladolid, 47011 Valladolid, Spain*

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Abstract

The quantum free particle on the sphere S_κ^2 ($\kappa > 0$) and on the hyperbolic plane H_κ^2 ($\kappa < 0$) is studied using a formalism that considers the curvature κ as a parameter. The first part is mainly concerned with the analysis of some geometric formalisms appropriate for the description of the dynamics on the spaces $(S_\kappa^2, \mathbb{R}^2, H_\kappa^2)$ and with the transition from the classical κ -dependent system to the quantum one using the quantization of the Noether momenta. The Schrödinger separability and the quantum superintegrability are also discussed. The second part is devoted to the resolution of the κ -dependent Schrödinger equation. First the characterization of the κ -dependent ‘curved’ plane waves is analyzed and then the specific properties of the spherical case are studied with great detail. It is proved that if $\kappa > 0$ then a discrete spectrum is obtained. The wavefunctions, that are related with a κ -dependent family of orthogonal polynomials, are explicitly obtained.

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^{a)} *E-mail address:* jfc@unizar.es

^{b)} *E-mail address:* mfran@unizar.es

^{c)} *E-mail address:* msn@fta.uva.es

1 Introduction

The correct formulation of quantum mechanics on spaces of constant curvature is a problem that can lead to important difficulties. There are some fundamental quantum questions, well stated in the Euclidean space, that become difficult to formulate on a curved space. The study of these questions is important, not only for extending our knowledge of certain fundamental points of quantum mechanics, but also because it is very convenient for the construction of more general relativistic theories [1],[2]. In addition, this matter has become also important for the study of certain questions arising in applied nonrelativistic quantum mechanics. We mention here two examples related with two dimensional quantum mechanics and with condensed matter physics. In the first case (motion of a particle on a two-dimensional surface) the existence of Landau levels for the motion of a charged particle under perpendicular magnetic fields has been also studied for the case of non-Euclidean geometries [3]-[6]. Concerning the second point, the study of quantum dots has also led to the use of models based in quantum mechanics in spaces of constant curvature [7]-[11].

The first step was probably given by Schrödinger who made use of a factorization method [12] for the study of the Hydrogen atom in a spherical geometry. Then Infeld [13] and Stevenson [14] studied the same system and Infeld and Schild [15] considered this problem in an open universe of constant negative curvature. Other more recent papers on the Hydrogen atom in a curved space are [16]-[18]. Other authors (see e.g. [19]-[22] and references therein) studied the quantum oscillator on curved spaces. We also mention that the path integral formulation has been also studied in curved spaces [23]-[24].

On the other side Higgs studied in 1979, but from the point of view of classical mechanics, the existence of dynamical symmetries in a spherical geometry [25]. In fact his study was mainly focussed on the existence of the spherical versions of the Runge-Lenz vector (Kepler) and the Fradkin tensor (harmonic oscillator). Since then a certain number of authors [26]-[46] have considered these questions or some other properties characterizing the Hamiltonian systems defined on curved spaces. We recall that the Kepler and the harmonic oscillator are two systems separable in several different coordinate systems and because of this they are superintegrable with quadratic constants of motion. It has been proved [47]-[54] the existence of other not so simple potentials (noncentral) that are also multiply separable on spaces with curvature.

We also mention the study of polygonal billiards (systems enclosed by arcs of geodesics) on surfaces with curvature [55],[56]; one of the main points is that some simple motions, that are integrable in the Euclidean case, can become ergodic when the curvature is negative. The quantum version of these systems leads to the study of chaoticity in quantum systems.

The present article is concerned with the study of the quantum free particle on spherical and hyperbolic spaces. This problem is usually considered as rather simple in the Euclidean case mainly because the solutions are plane-wave states that are in fact momentum eigenfunctions, that is, eigenstates of the linear momentum operator. The plane waves are thus simultaneous eigenfunctions of energy and linear momentum. Nevertheless the situation is much more complicate in a space with curvature mainly for two reasons. Firstly because the canonical momenta p_i do not coincide with the Noether momenta P_i . Secondly because the Noether momenta do

not Poisson commute (Classical mechanics) and the corresponding self-adjoint quantum versions \hat{P}_i do not commute as operators. We can also add to these two points that a plane-wave is an Euclidean concept; therefore the meaning of plane-wave in a curved space is not clear and a new more general definition must be introduced.

The main goal of this paper is to solve the problem by obtaining all the results making use of a curvature dependent approach. In fact, one of the main characteristics of this paper is that it consider the curvature κ as a parameter; that is, it presents all the mathematical expressions in a κ -dependent way. In fact, this κ -dependent approach was already used in some previous related classical [36, 37, 57] (see also [58]) and quantum studies [59]-[61].

We begin the paper with the analysis of some κ -dependent geometric formalisms appropriate for the description of the dynamics on the spaces $(S_\kappa^2, \mathbb{R}^2, H_\kappa^2)$ with constant curvature κ and this is done according with the study carried on in Ref. [60]. The first sections present a joint approach to both spherical ($\kappa > 0$) and hyperbolic dynamics ($\kappa < 0$) in such a way that the standard Euclidean dynamics just appears as the particular $\kappa = 0$ case. Then, the more specific properties are studied with detail but in separate sections. After the first introductory paragraphs, the rest of the article is mainly concerned with the following two points:

- Transition from the classical κ -dependent system to the quantum one using as an approach the quantization of the Noether momenta.
- Exact resolution of the κ -dependent Schrödinger equation, κ -dependent plane waves, families of new κ -dependent orthogonal polynomials, and existence of bound states.

It is interesting to remark that the curvature κ introduce some new coefficients in the kinetic term so that the problem of quantizing a system defined in a space with constant curvature can be related with the problem of quantizing a system with a position-dependent mass.

In more detail, the plan of the article is as follows: In Sec. 2 we study the classical system, the quantization and the separability of the Schrödinger equation. In Sec. 3 we discuss the existence of another geometric description. Sec. 4 is devoted to the spherical $\kappa > 0$ case and Sec. 5 to the analysis of the eigenfunctions $\Psi_{m,n}$ and energies $E_{m,n}$. The hyperbolic $\kappa < 0$ case is studied in Sec. 6. In Sec. 7 we briefly analyze the existence of an alternative approach and its relation with the presence of the angular momentum. Finally, in Sec. 8 we make some final comments.

2 Spaces of constant curvature, κ -dependent formalism and quantization

In what follows, all the mathematical expressions will depend of the curvature κ as a parameter, in such a way that for $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$, we will obtain the corresponding property particularized for the dynamical system on the sphere, on the Euclidean plane, or on the hyperbolic plane respectively.

The relations among several different possible approaches to the Lagrangian or Hamiltonian dynamics on spaces with curvature are discussed with a certain detail in [60, 61]; next, we summarize in the following points the relation between the approach presented in this paper and the Higgs approach. First, we recall that the three spaces with constant curvature, sphere S_κ^2 ($\kappa > 0$), Euclidean plane \mathbb{E}^2 , and hyperbolic plane H_κ^2 ($\kappa < 0$), can be considered as three different situations inside a family of Riemannian manifolds $M_\kappa^2 = (S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ with the curvature $\kappa \in \mathbb{R}$ as a parameter (it seems that this geometric idea was first introduced by Weierstrass and Killing [58]). In fact, if we make use of the following κ -dependent trigonometric (hyperbolic) functions

$$C_\kappa(x) = \begin{cases} \cos \sqrt{\kappa} x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh \sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases} \quad S_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \text{if } \kappa < 0, \end{cases}$$

and

$$T_\kappa(x) = \frac{S_\kappa(x)}{C_\kappa(x)},$$

then the expression of the differential arc length element in geodesic polar coordinates (ρ, ϕ) on M_κ^2 can be written as follows

$$ds_\kappa^2 = d\rho^2 + S_\kappa^2(\rho) d\phi^2,$$

so it reduces to

$$ds_1^2 = d\rho^2 + (\sin^2 \rho) d\phi^2, \quad ds_0^2 = d\rho^2 + \rho^2 d\phi^2, \quad ds_{-1}^2 = d\rho^2 + (\sinh^2 \rho) d\phi^2,$$

in the three particular cases of the unit sphere, the Euclidean plane, and the ‘unit’ Lobachewski plane (Note that ρ denotes the distance along a geodesic on the manifold M_κ^2 and not the radius of a sphere). If we make use of this formalism then the Lagrangian of the geodesic motion (free particle) on M_κ^2 is given by [36, 37, 61]

$$\mathbb{L}(\kappa) = \left(\frac{1}{2}\right) (v_\rho^2 + S_\kappa^2(\rho) v_\phi^2) \quad (1)$$

1. If we consider the κ -dependent change $\rho \rightarrow r = S_\kappa(\rho)$ then the Lagrangian $\mathbb{L}(\kappa)$ becomes

$$L(\kappa) = \frac{1}{2} \left(\frac{v_r^2}{1 - \kappa r^2} + r^2 v_\phi^2 \right)$$

and, if we change to Cartesian coordinates, we arrive to

$$L(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left[v_x^2 + v_y^2 - \kappa (xv_y - yv_x)^2 \right], \quad r^2 = x^2 + y^2.$$

This function is just the Lagrangian studied in Ref. [57] at the classical level and in [59]-[61] at the quantum level (it can also be obtained as the two-dimensional version of the kinetic term of the one-dimensional Lagrangian $L(x, v_x; \lambda)$ for the nonlinear equation of Mathews and Lakshmanan [62, 63]).

2. If we consider the κ -dependent change $\rho \rightarrow r' = T_\kappa(\rho)$ then the Lagrangian $\mathbb{L}(\kappa)$ becomes

$$L_H(\kappa) = \frac{1}{2} \left(\frac{v_r'^2}{(1 + \kappa r'^2)^2} + \frac{r'^2 v_\phi^2}{(1 + \kappa r'^2)} \right),$$

and, if we change to Cartesian coordinates, we arrive to

$$L_H(\kappa) = \frac{1}{2} \frac{1}{(1 + \kappa r'^2)^2} \left[v_x^2 + v_y^2 + \kappa (xv_y - yv_x)^2 \right], \quad r'^2 = x^2 + y^2,$$

that coincides with the kinetic term of the Lagrangian introduced by Higgs in Ref. [25] and studied later on by other authors [26]-[31] (the study of Higgs was originally limited to a spherical geometry but the idea can also be applied to the hyperbolic space).

Hence the three Lagrangians, $\mathbb{L}(\kappa)$, $L(\kappa)$ and $L_H(\kappa)$, are related by diffeomorphisms, must be considered as dynamically equivalent, and they can be alternatively used as a starting point for the construction of the Hamiltonian quantum system. In the following we will make use of the Hamiltonian dynamics determined by the κ -dependent Lagrangian denoted by $L(\kappa)$ with coordinates (x, y) .

At this point we make the following observation. It is frequent to present the geometric approach to the hyperbolic plane in two steps: (i) First, consider a two dimensional pseudosphere (the upper sheet of a two-sheeted hyperboloid of revolution) inside a three-dimensional space with Minkowskian metric. (ii) Then the two-dimensional model of the hyperbolic space is obtained by projection on the two-dimensional plane. The approach presented in this paper is more direct and intrinsic (in differential geometric terms) and it presents directly the hyperbolic space as the manifold $M_\kappa^2 = (S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ endowed with the appropriate κ -dependent metric.

2.1 Killing vectors, Noether symmetries and Noether momenta

We start with the following expression for the differential element of distance on the family $M_\kappa^2 = (S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$

$$ds_\kappa^2 = \left(\frac{1}{1 - \kappa r^2} \right) \left[(1 - \kappa y^2) dx^2 + (1 - \kappa x^2) dy^2 + 2\kappa xy dx dy \right], \quad r^2 = x^2 + y^2, \quad (2)$$

that can also be written as

$$ds_\kappa^2 = \left(\frac{1}{1 - \kappa r^2} \right) \left[dx^2 + dy^2 - \kappa (x dy - y dx)^2 \right]. \quad (3)$$

Then the following three vector fields

$$X_1(\kappa) = \sqrt{1 - \kappa r^2} \frac{\partial}{\partial x}, \quad X_2(\kappa) = \sqrt{1 - \kappa r^2} \frac{\partial}{\partial y}, \quad X_J = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

are Killing vector fields, that is, infinitesimal generators of isometries of the κ -dependent metric ds_κ^2 . The Lie brackets of these vector fields are given by

$$[X_1(\kappa), X_2(\kappa)] = \kappa X_J, \quad [X_1(\kappa), X_J] = -X_2, \quad [X_2(\kappa), X_J] = X_1,$$

so that they close, depending of the sign of κ , the Lie algebra of the group of isometries of the Spherical, Euclidean and Hyperbolic spaces. Notice that only when $\kappa = 0$ (Euclidean space) X_1 and X_2 commute.

The geodesic motion on $M_\kappa^2 = (S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ is determined by a Lagrangian L reduced to the κ -dependent kinetic term $T(\kappa)$ without any potential

$$L = T(\kappa) = \frac{1}{2} \left(\frac{1}{1 - \kappa r^2} \right) \left[v_x^2 + v_y^2 - \kappa (xv_y - yv_x)^2 \right], \quad r^2 = x^2 + y^2, \quad (4)$$

where the parameter κ can take both positive and negative values; of course it is clear that in the spherical case as $\kappa > 0$, the function (and the associated dynamics) will have a singularity at $1 - \kappa r^2 = 0$; so in this case we shall restrict the study of the dynamics to the region $r^2 < 1/\kappa$ where the kinetic energy function is positive definite. This free-particle Lagrangian is invariant under the action of the three vector fields $X_1(\kappa)$, $X_2(\kappa)$, and X_J , in the sense that, if we denote by X_r^t , $r = 1, 2, J$, the natural lift to the tangent bundle (phase space $\mathbb{R}^2 \times \mathbb{R}^2$) of the vector field X_r ,

$$\begin{aligned} X_1^t(\kappa) &= \sqrt{1 - \kappa r^2} \frac{\partial}{\partial x} - \kappa \left(\frac{xv_x + yv_y}{\sqrt{1 - \kappa r^2}} \right) \frac{\partial}{\partial v_x}, \\ X_2^t(\kappa) &= \sqrt{1 - \kappa r^2} \frac{\partial}{\partial y} - \kappa \left(\frac{xv_x + yv_y}{\sqrt{1 - \kappa r^2}} \right) \frac{\partial}{\partial v_y}, \\ X_J^t &= x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v_y} - v_y \frac{\partial}{\partial v_x}, \end{aligned}$$

then the Lie derivatives of $T(\kappa)$ with respect to $X_r^t(\kappa)$ vanish, that is

$$X_r^t(\kappa) \left(T(\kappa) \right) = 0, \quad r = 1, 2, J.$$

They represent three exact Noether symmetries for the geodesic motion. If we denote by θ_L the Lagrangian one-form

$$\theta_L = \left(\frac{v_x + \kappa Jy}{1 - \kappa r^2} \right) dx + \left(\frac{v_y - \kappa Jx}{1 - \kappa r^2} \right) dy,$$

then the associated Noether constants of the motion $P_1(\kappa)$, $P_2(\kappa)$ and J , are given by

$$\begin{aligned} P_1(\kappa) &= i(X_1^t(\kappa)) \theta_L = \frac{v_x + \kappa Jy}{\sqrt{1 - \kappa r^2}} \\ P_2(\kappa) &= i(X_2^t(\kappa)) \theta_L = \frac{v_y - \kappa Jx}{\sqrt{1 - \kappa r^2}} \\ J &= i(X_J^t) \theta_L = xv_y - yv_x \end{aligned}$$

2.2 κ -dependent Hamiltonian and Quantization

The Legendre transformation leads to the following expression for the κ -dependent Hamiltonian

$$H(\kappa) = \left(\frac{1}{2} \right) \left[p_x^2 + p_y^2 - \kappa (xp_x + yp_y)^2 \right]. \quad (5)$$

The three Noether momenta become

$$P_1(\kappa) = \sqrt{1 - \kappa r^2} p_x, \quad P_2(\kappa) = \sqrt{1 - \kappa r^2} p_y, \quad J = xp_y - yp_x,$$

with Poisson brackets

$$\{P_1(\kappa), P_2(\kappa)\} = \kappa J, \quad \{P_1(\kappa), J\} = -P_2(\kappa), \quad \{P_2(\kappa), J\} = P_1(\kappa),$$

and such that

$$\{P_1(\kappa), H(\kappa)\} = 0, \quad \{P_2(\kappa), H(\kappa)\} = 0, \quad \{J, H(\kappa)\} = 0.$$

A very important property is that the Hamiltonian can be written as a function of the three Noether momenta. It is given by

$$H(\kappa) = \left(\frac{1}{2}\right) [P_1^2 + P_2^2 + \kappa J^2]. \quad (6)$$

We note that $H(\kappa)$ is just the Casimir of the above Poisson algebra.

The following task is to find the appropriate quantum mechanical Hilbert space and this means to obtain a measure that reduces to the standard one when $\kappa \rightarrow 0$. An important property is that the only measure on the space \mathbb{R}^2 , with coordinates (x, y) , that is invariant under the action of the three vector fields $X_1(\kappa)$, $X_2(\kappa)$, and X_J , is given by

$$d\mu_\kappa = \left(\frac{1}{\sqrt{1 - \kappa r^2}}\right) dx dy,$$

up to a constant factor (see Ref. [60] for a proof).

This property means that the quantum Hamiltonian must be self-adjoint, not in the standard space $L^2(\mathbb{R}^2)$, but in the Hilbert space $L_\kappa^2(d\mu_\kappa)$ defined as

- (i) In the hyperbolic $\kappa < 0$ case, the space $L_\kappa^2(d\mu_\kappa)$ is $L^2(\mathbb{R}^2, d\mu_\kappa)$.
- (ii) In the spherical $\kappa > 0$ case, the space $L_\kappa^2(d\mu_\kappa)$ is $L_0^2(\mathbb{R}_\kappa^2, d\mu_\kappa)$ where \mathbb{R}_κ^2 denotes the region $r^2 \leq 1/\kappa$ and the subscript means that the functions must vanish at the boundary of this region

(the question of the boundary conditions will be discussed with more detail when considering the Sturm-Liouville problem). For obtaining the expression of the operator $\hat{H}(\kappa)$ we first consider the operators \hat{P}_1 and \hat{P}_2 , representing the quantum version of the Noether momenta P_1 and P_2 , that must be also self-adjoint in the space $L_\kappa^2(d\mu_\kappa)$. They are given by

$$\begin{aligned} \hat{P}_1 &= -i\hbar \sqrt{1 - \kappa r^2} \frac{\partial}{\partial x}, \\ \hat{P}_2 &= -i\hbar \sqrt{1 - \kappa r^2} \frac{\partial}{\partial y}. \end{aligned}$$

Then we arrive to the following correspondence

$$\begin{aligned} P_1^2 &\rightarrow -\hbar^2 \left(\sqrt{1 - \kappa r^2} \frac{\partial}{\partial x} \right) \left(\sqrt{1 - \kappa r^2} \frac{\partial}{\partial x} \right), \\ P_2^2 &\rightarrow -\hbar^2 \left(\sqrt{1 - \kappa r^2} \frac{\partial}{\partial y} \right) \left(\sqrt{1 - \kappa r^2} \frac{\partial}{\partial y} \right), \end{aligned}$$

in such a way that the quantum Hamiltonian operator $\hat{H}(\kappa)$

$$\hat{H}(\kappa) = \left(\frac{1}{2} \right) [\widehat{P}_1^2 + \widehat{P}_2^2 + \kappa \hat{J}^2]$$

is given by

$$\begin{aligned} \hat{H}(\kappa) = & - \frac{\hbar^2}{2m} \left[(1 - \kappa r^2) \frac{\partial^2}{\partial x^2} - \kappa x \frac{\partial}{\partial x} \right] - \frac{\hbar^2}{2m} \left[(1 - \kappa r^2) \frac{\partial^2}{\partial y^2} - \kappa y \frac{\partial}{\partial y} \right] \\ & - \kappa \frac{\hbar^2}{2m} \left[x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right] \end{aligned} \quad (7)$$

The first important property of this Hamiltonian is that it admits the following decomposition

$$\hat{H}(\kappa) = \widehat{H}_1 + \widehat{H}_2 + \kappa \hat{J}^2,$$

where the three partial operators \widehat{H}_1 , \widehat{H}_2 y \hat{J}^2 are respectively given by

$$\begin{aligned} \widehat{H}_1(\kappa) &= -\frac{\hbar^2}{2m} \left[(1 - \kappa r^2) \frac{\partial^2}{\partial x^2} - \kappa x \frac{\partial}{\partial x} \right] \\ \widehat{H}_2(\kappa) &= -\frac{\hbar^2}{2m} \left[(1 - \kappa r^2) \frac{\partial^2}{\partial y^2} - \kappa y \frac{\partial}{\partial y} \right] \\ \hat{J}^2 &= -\frac{\hbar^2}{2m} \left[x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right] \end{aligned}$$

in such a way that the total Hamiltonian \hat{H} commutes, for any value of the parameter κ , with each one of the three partial terms

$$[\hat{H}(\kappa), \widehat{H}_1(\kappa)] = 0, \quad [\hat{H}(\kappa), \widehat{H}_2(\kappa)] = 0, \quad [\hat{H}(\kappa), \hat{J}^2] = 0.$$

The vanishing of these three commutators means that the κ -dependent Hamiltonian (7) describes a quantum superintegrable system [64]-[73]. This property was well known in the Euclidean $\kappa = 0$ case but now it appears in a different form because of presence of the term $\kappa \hat{J}^2$.

2.3 Schrödinger equation and Separability

Now, if we consider the Schrödinger equation

$$\hat{H} \Psi = E \Psi, \quad (8)$$

as we have the following property

$$[\widehat{H}_1, \widehat{H}_2 + \kappa \hat{J}^2] = 0, \quad [\widehat{H}_1 + \kappa \hat{J}^2, \widehat{H}_2] = 0, \quad [\widehat{H}_1 + \widehat{H}_2, \hat{J}^2] = 0,$$

then, we have three different sets of compatible observables and therefore three different ways of obtaining a Hilbert basis of common eigenstates.

1. The two operators \widehat{H}_1 and $\widehat{H}_2 + \kappa \widehat{J}^2$ are a (complete) set of commuting observables; therefore they represent two quantities that can be simultaneously measured. Thus, the first way of looking for Ψ is as a solution of the following two equations

$$\widehat{H}_1 \Psi = e_1 \Psi, \quad (\widehat{H}_2 + \kappa \widehat{J}^2) \Psi = e_{2j} \Psi.$$

In this case the total energy is given by $E = e_1 + e_{2j}$ and the associated wave function can be denoted by $\Psi(e_1, e_{2j})$.

2. The two operators $\widehat{H}_1 + \kappa \widehat{J}^2$ and \widehat{H}_2 are a (complete) set of commuting observables. Thus, the second way of looking for Ψ is as a solution of the following two equations

$$(\widehat{H}_1 + \kappa \widehat{J}^2) \Psi = e_{1j} \Psi, \quad \widehat{H}_2 \Psi = e_2 \Psi.$$

In this case we have $E = e_{1j} + e_2$ and Ψ can be denoted by $\Psi(e_{1j}, e_2)$.

3. The third (complete) set of commuting observables is provided by $\widehat{H}_1 + \widehat{H}_2$ and \widehat{J}^2 . So in this case we have

$$(\widehat{H}_1 + \widehat{H}_2) \Psi = e_{12} \Psi, \quad \widehat{J}^2 \Psi = e_j \Psi.$$

Thus, the two physically measurable quantities are e_{12} and the angular momentum j , the total energy is given by $E = e_{12} + \kappa e_j$ and the wave function so defined can be denoted by $\Psi(e_{12}, e_j)$.

The existence of these three alternative descriptions arises from the presence of the term $\kappa \widehat{J}^2$ inside the kinetic part of the Hamiltonian. Notice that the second approach can be considered as symmetric to the first one. Nevertheless, although they are closely related, they lead however to different solutions with different properties; that is, $\Psi(e_1, e_{2j}) \neq \Psi(e_{1j}, e_2)$. This fact is a consequence of the nonlinear character of the model since in the linear limit, when $\kappa \rightarrow 0$, then both descriptions coincide.

The κ -dependent metric ds_κ^2 is not diagonal in the coordinates (x, y) and the Schrödinger equation (8) is not separable in these coordinates because of the κ -dependent term. Nevertheless, the classical Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 - \kappa \left(x \frac{\partial S}{\partial x} + y \frac{\partial S}{\partial y}\right)^2 = 0$$

and the quantum Schrödinger equation admit separability in the following three different orthogonal coordinate systems:

1. κ -dependent coordinates (z_x, y) with z_x defined by $z_x = x/\sqrt{1 - \kappa y^2}$.
2. κ -dependent coordinates (x, z_y) with z_y defined by $z_y = y/\sqrt{1 - \kappa x^2}$.
3. Polar coordinates (r, ϕ) .

At this point we recall that the existence of multiple separability is a property directly related with superintegrability (in fact with quadratic superintegrability).

Next we start our study with the first coordinate system. First note that the change $(x, y) \rightarrow (z_x, y)$ transforms, in the $k > 0$ spherical case, the circular domain $x^2 + y^2 < 1/\kappa$ into the square region $z_x^2 < 1/\kappa$, $y^2 < 1/\kappa$, in the (z_x, y) plane.

Using (z_x, y) coordinates the two partial Hamiltonians become:

- (i) The Hamiltonian \hat{H}_1 is given by

$$\hat{H}_1 = -\frac{\hbar^2}{2m} \tilde{H}_1, \quad \tilde{H}_1 = (1 - \kappa z_x^2) \frac{\partial^2}{\partial z_x^2} - (\kappa z_x) \frac{\partial}{\partial z_x}.$$

- (ii) The Hamiltonian $\hat{H}_2 + \kappa \hat{J}^2$ given by

$$\hat{H}_2 + \kappa \hat{J}^2 = -\frac{\hbar^2}{2m} [\tilde{H}_2 + \kappa \tilde{J}^2],$$

is represented by the following differential operator

$$\tilde{H}_2 + \kappa \tilde{J}^2 = \frac{\kappa y^2}{1 - \kappa y^2} \left[(1 - \kappa z_x^2) \frac{\partial^2}{\partial z_x^2} - (\kappa z_x) \frac{\partial}{\partial z_x} \right] + \left[(1 - \kappa y^2) \frac{\partial^2}{\partial y^2} - (2\kappa y) \frac{\partial}{\partial y} \right],$$

that can be rewritten as follows

$$\tilde{H}_2 + \kappa \tilde{J}^2 = \frac{\kappa y^2}{1 - \kappa y^2} \tilde{H}_1 + \left[(1 - \kappa y^2) \frac{\partial^2}{\partial y^2} - (2\kappa y) \frac{\partial}{\partial y} \right].$$

Consequently the κ -dependent Schrödinger equation is in fact separable in the (z_x, y) coordinates. Thus the two-dimensional problem has been decoupled in two one-dimensional equations.

- (i) The Schrödinger equation $\hat{H}_1 \Psi = e_1 \Psi$ for the first partial Hamiltonian \hat{H}_1 leads to the following equation with derivatives with respect to the variable z_x alone

$$(1 - \kappa z_x^2) \Psi''_{z_x z_x} - (\kappa z_x) \Psi'_{z_x} + \mu \Psi = 0, \quad \mu = \left(\frac{2m}{\hbar^2}\right) e_1.$$

- (ii) The Schrödinger equation $(\hat{H}_2 + \kappa \hat{J}^2) \Psi = e_{2j} \Psi$ for the second partial Hamiltonian $\hat{H}_2 + \kappa \hat{J}^2$ leads to the following μ -dependent equation with derivatives with respect to the variable y alone

$$-\frac{\kappa y^2}{1 - \kappa y^2} (\mu \Psi) + \left[(1 - \kappa y^2) \Psi''_{yy} - (2\kappa y) \Psi'_y \right] + \nu \Psi = 0, \quad \nu = \left(\frac{2m}{\hbar^2}\right) e_{2j}.$$

Thus, if we assume that $\Psi(z_x, y)$ is a function of the form

$$\Psi(z_x, y) = Z(z_x) Y(y),$$

then we arrive to

$$(1 - \kappa z_x^2) Z'' - (\kappa z_x) Z' + \mu Z = 0,$$

and

$$(1 - \kappa y^2) Y'' - (2\kappa y) Y' - \mu \kappa \left(\frac{y^2}{1 - \kappa y^2} \right) Y + \nu Y = 0.$$

3 An alternative approach: Parallel geodesic coordinates

The study presented in Sec. 2 (symmetries, quantization, separation of variables) was constructed starting with the expression (2) for ds_κ^2 ; nevertheless, it can alternatively be developed in other coordinate systems. Now we present in this section the results obtained when making use of parallel geodesic coordinates (u, y_κ) (see the Appendix for more information on the geometric origin of this particular system of coordinates and Ref. [36, 37] (and references therein) for some papers that make use of this formalism).

The following expression written in (u, y_κ) parallel coordinates

$$ds_\kappa^2 = C_\kappa^2(y_\kappa) du^2 + dy_\kappa^2. \quad (9)$$

represents the differential element of distance on the spaces $(S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ with constant curvature κ . So a standard lagrangian (kinetic term minus a potential function) has the following form

$$L(\kappa) = \left(\frac{1}{2}\right)(C_\kappa^2(y_\kappa) v_u^2 + v_{y_\kappa}^2) - U(u, y_\kappa, \kappa),$$

in such a way that the Euclidean system is just given by the particular value of $L(\kappa)$ in $\kappa = 0$

$$\lim_{\kappa \rightarrow 0} L(\kappa) = \left(\frac{1}{2}\right)(v_x^2 + v_y^2) - V(x, y), \quad V(x, y) = U(x, y_\kappa, \kappa) \Big|_{\kappa=0}.$$

The three κ -dependent Killing vector, Y_1 , $Y_2(\kappa)$, and $Y_J(\kappa)$, have now the following expressions in parallel coordinates

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial u}, \\ Y_2(\kappa) &= \kappa S_\kappa(u) T_\kappa(y_\kappa) \frac{\partial}{\partial u} + C_\kappa(u) \frac{\partial}{\partial y_\kappa}, \\ Y_J(\kappa) &= C_\kappa(u) T_\kappa(y_\kappa) \frac{\partial}{\partial u} - S_\kappa(u) \frac{\partial}{\partial y_\kappa}, \end{aligned}$$

(Y_1 is now κ -independent) and the associated linear constants of motion are given by

$$\begin{aligned} P_1(\kappa) &= C_\kappa^2(y_\kappa) v_u, \\ P_2(\kappa) &= \kappa S_\kappa(u) C_\kappa(y_\kappa) S_\kappa(y_\kappa) v_u + C_\kappa(u) v_{y_\kappa}, \\ J(\kappa) &= C_\kappa(u) C_\kappa(y_\kappa) S_\kappa(y_\kappa) v_u - S_\kappa(u) v_{y_\kappa}. \end{aligned}$$

Then, when moving to the Hamiltonian formalism we obtain that the Noether momenta are given by

$$\begin{aligned} P_1(\kappa) &= p_u, \\ P_2(\kappa) &= \kappa S_\kappa(u) T_\kappa(y_\kappa) p_u + C_\kappa(u) p_{y_\kappa}, \\ J(\kappa) &= C_\kappa(u) T_\kappa(y_\kappa) p_u - S_\kappa(u) p_{y_\kappa}, \end{aligned}$$

in such a way that the geodesic Hamiltonian

$$H(\kappa) = \left(\frac{1}{2}\right) \left(\frac{p_u^2}{C_\kappa^2(y_\kappa)} + p_{y_\kappa}^2 \right),$$

can also be written as

$$H(\kappa) = \left(\frac{1}{2}\right) (P_1^2 + P_2^2 + \kappa J^2).$$

The κ -dependent measure $d\mu_\kappa$, invariant under the action of the three vector fields Y_1 , Y_2 and Y_J , is given by

$$d\mu_\kappa = C_\kappa(y_\kappa) du dy_\kappa,$$

and the transition from classical to quantum mechanics via the Noether momenta is now represented by the following correspondence

$$\begin{aligned} P_1 &\rightarrow \widehat{P}_1 = -i\hbar \frac{\partial}{\partial u}, \\ P_2 &\rightarrow \widehat{P}_2 = -i\hbar \left(\kappa S_\kappa(u) T_\kappa(y_\kappa) \frac{\partial}{\partial u} + C_\kappa(u) \frac{\partial}{\partial y_\kappa} \right), \\ J &\rightarrow \widehat{J} = -i\hbar \left(C_\kappa(u) T_\kappa(y_\kappa) \frac{\partial}{\partial u} - S_\kappa(u) \frac{\partial}{\partial y_\kappa} \right), \end{aligned}$$

so that we arrive to

1. The quantum operator \widehat{H}_1 is given by

$$\widehat{H}_1 = -\frac{\hbar^2}{2m} \widetilde{H}_1, \quad \widetilde{H}_1 = \frac{\partial^2}{\partial u^2}.$$

2. The quantum operator $\widehat{H}_2 + \kappa \widehat{J}^2$ given by

$$\widehat{H}_2 + \kappa \widehat{J}^2 = -\frac{\hbar^2}{2m} [\widetilde{H}_2 + \kappa \widetilde{J}^2]$$

is represented by the following differential operator

$$\widetilde{H}_2 + \kappa \widetilde{J}^2 = \kappa T_\kappa^2(y_\kappa) \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial y_\kappa^2} - \kappa T_\kappa(y_\kappa) \frac{\partial}{\partial y_\kappa},$$

that can be rewritten as follows

$$\widetilde{H}_2 + \kappa \widetilde{J}^2 = \kappa T_\kappa^2(y_\kappa) \widetilde{H}_1 + \left[\frac{\partial^2}{\partial y_\kappa^2} - \kappa T_\kappa(y_\kappa) \frac{\partial}{\partial y_\kappa} \right].$$

In this way, the two Schrödinger equations for \widehat{H}_1 (equation for the variable u) and for $\widehat{H}_2 + \kappa \widehat{J}^2$ (equation for the variable y_κ) are

(i) The Schrödinger equation $\tilde{H}_1 \psi = \mu \Psi$ determined by the Hamiltonian \hat{H}_1 leads to

$$\Psi''_{uu} + \mu \Psi = 0, \quad \mu = \left(\frac{2m}{\hbar^2}\right) e_1.$$

(ii) The Schrödinger equation $(\tilde{H}_2 + \kappa \tilde{J}^2) \Psi = \nu \Psi$ determined by the Hamiltonian $\hat{H}_2 + \kappa \hat{J}^2$ leads to

$$-\kappa \mu T_\kappa^2(y_\kappa) \Psi + \left[\Psi''_{y_\kappa y_\kappa} - \kappa T_\kappa(y_\kappa) \Psi'_{y_\kappa} \right] + \nu \Psi = 0, \quad \nu = \left(\frac{2m}{\hbar^2}\right) e_{2j}.$$

Thus, if we assume that $\Psi(u, y_\kappa)$ is a function of the form

$$\Psi(u, y_\kappa) = U(u) Y(y_\kappa),$$

then we arrive to

$$U''_{uu} + \mu U = 0,$$

and

$$Y''_{y_\kappa y_\kappa} - \kappa T_\kappa(y_\kappa) Y'_{y_\kappa} + \left(\nu - \kappa \mu T_\kappa^2(y_\kappa) \right) Y = 0.$$

The U -equation is just the same equation that in the Euclidean case (so the solution is a u -plane-wave, that is, a plane-wave along the geodesic curve $y_\kappa = 0$). Concerning the Y -equation, it can be simplified by using the following factorization

$$Y = (C_\kappa(y_\kappa))^{g_\kappa} p(y_\kappa), \quad g_\kappa = \sqrt{\mu/\kappa}, \quad \kappa > 0,$$

so that we arrive to

$$p''_{y_\kappa y_\kappa} - \kappa (1 + 2g_\kappa) T_\kappa(y_\kappa) p'_{y_\kappa} + (\nu - \kappa g_\kappa) p = 0.$$

4 Spherical $\kappa > 0$ case

Now we return to the approach developed in Sec. 2 and study the spherical $\kappa > 0$ case.

4.1 Resolution of the Z -equation

The first equation to be solved is

$$(1 - \kappa z_x^2) Z'' - (\kappa z_x) Z' + \mu Z = 0, \quad \kappa > 0. \quad (10)$$

This equation coincides (up to the appropriate changes of notation) with the equation corresponding to a one-dimensional κ -dependent free particle. Assuming for Z an expression of the form

$$Z = e^{i u(z_x)},$$

with $u(z_x)$ a function to be determined, then we obtain that the general solution of (10) is given by

$$Z = A e^{i u(z_x)} + B e^{-i u(z_x)}, \quad u = \frac{\sqrt{\mu}}{\sqrt{\kappa}} \arcsin(\sqrt{\kappa} z_x), \quad \kappa > 0, \quad (11)$$

that is a well defined function for all the values of z_x . This solution satisfies the appropriate Euclidean limit

$$\lim_{\kappa \rightarrow 0} Z = A e^{i k_x x} + B e^{-i k_x x}, \quad k_x = \sqrt{\mu},$$

and therefore it can be considered as representing a κ -dependent curved plane-wave or a κ -dependent deformation of the Euclidean plane-wave solution.

4.2 Resolution of the Y -equation

The second equation to be solved is

$$(1 - \kappa y^2) Y'' - (2\kappa y) Y' - \mu\kappa \left(\frac{y^2}{1 - \kappa y^2} \right) Y + \nu Y = 0, \quad \kappa > 0, \quad (12)$$

that, although it has certain similarity with the Eq. (10), it does not coincide with it (two differences: the factor 2 in the coefficient of Y' and the rational μ -dependent term). The main reason for this asymmetry is that, when introducing separability in the Schrödinger equation, the angular momentum term \hat{J}^2 was displaced into this second equation.

It can be verified that the function Ψ_Y defined by

$$\Psi_Y = (1 - \kappa y^2)^{(1/2)g_\kappa}, \quad g_\kappa = \sqrt{\frac{\mu}{\kappa}},$$

satisfies the following property

$$\left[(1 - \kappa y^2) \frac{d^2}{dy^2} - (2\kappa y) \frac{d}{dy} - \mu\kappa \left(\frac{y^2}{1 - \kappa y^2} \right) \right] \Psi_Y = -\kappa g_\kappa \Psi_Y.$$

Thus Ψ_Y represents the exact solution in the very particular case of $\nu = \kappa g_\kappa$. This property suggests the following factorization for the function $Y(y)$

$$Y(y) = p(y) (1 - \kappa y^2)^{(1/2)g_\kappa}, \quad g_\kappa = \sqrt{\frac{\mu}{\kappa}},$$

where the factor on the right satisfies the following Euclidean limit

$$\lim_{\kappa \rightarrow 0} (1 - \kappa y^2)^{(1/2)g_\kappa} = \exp \left[\lim_{\kappa \rightarrow 0} \frac{\sqrt{\mu}}{2} \frac{\log(1 - \kappa y^2)}{\sqrt{\kappa}} \right] = \exp \left[\frac{\sqrt{\mu}}{2} \lim_{\kappa \rightarrow 0} \frac{-y^2}{(1 - \kappa y^2)} (2\sqrt{\kappa}) \right] = 1.$$

Then the equation becomes

$$(1 - \kappa y^2) p'' - 2\kappa (1 + g_\kappa) y p' + (\nu - \kappa g_\kappa) p = 0, \quad p = p(y). \quad (13)$$

This equation is an equation of hypergeometric type and it can be reduced to the canonical form of a hypergeometric Gauss equation with singular points in $w = 0$ and $w = 1$

$$w(1-w)p''_{ww} + (\lambda_a + \lambda_b w)p'_w + \lambda_c p = 0,$$

by making use of the change $y \rightarrow w$ given by

$$w = \frac{1}{2}(1 + \sqrt{k} y).$$

Nevertheless, as $y = 0$ is an ordinary point, it can be also directly solved by assuming a power expansion for the solution

$$p(y, \kappa) = \sum_{n=0}^{\infty} p_n(\kappa) y^n = p_0(\kappa) + p_1(\kappa) y + p_2(\kappa) y^2 + \dots$$

that leads to the following κ -dependent recursion relation

$$p_{n+2} = \left[\frac{\kappa n(n-1) + 2\kappa(1+g_\kappa)n - (\nu - \kappa g_\kappa)}{(n+2)(n+1)} \right] p_n.$$

Note that this relation shows that, as in the particular $\kappa = 0$ case, even power coefficients are related among themselves and the same is true for odd power coefficients. In both cases, having in mind that

$$\lim_{n \rightarrow \infty} \left| \frac{p_{n+2} y^{n+2}}{p_n y^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\kappa n(n-1) + 2\kappa(1+g_\kappa)n - (\nu - \kappa g_\kappa)}{(n+2)(n+1)} \right| |y^2| = |\kappa| |y^2|,$$

the radius of convergence R is given by $R = 1/\sqrt{|\kappa|}$. Hence, when we consider the limit $\kappa \rightarrow 0$, we recover the radius $R = \infty$ of the Euclidean equation.

The general solution is given by a linear combination

$$Y(\kappa) = (C Y_{ev}(y) + D Y_{od}(y)) (1 - \kappa y^2)^{(1/2)g_\kappa},$$

where $Y_{ev}(y)$ is an even function and $Y_{od}(y)$ is an odd function with $Y_{ev}(0) = 1$ and $Y'_{od}(0) = 1$. In the Euclidean limit, if c_n denotes $c_n = \lim_{\kappa \rightarrow 0} p_n(\kappa, \mu, \nu)$, then the recursion relation reduces to

$$c_{n+2} = \frac{(-\nu) c_n}{(n+2)(n+1)},$$

and the solution becomes

$$\lim_{\kappa \rightarrow 0} Y(\kappa) = C \cos(\sqrt{\nu} y) + D \sin(\sqrt{\nu} y),$$

that can also be written as

$$\lim_{\kappa \rightarrow 0} Y(\kappa) = \tilde{C} e^{i k_y y} + \tilde{D} e^{-i k_y y}, \quad k_y = \sqrt{\nu}.$$

and represents Euclidean plane-waves.

In the very particular case of an integer n such that

$$\nu - \kappa g_\kappa = \kappa n(n-1) + 2\kappa(1+g_\kappa)n$$

then we have $c_n \neq 0$, $c_{n+2} = 0$, and one of the two solutions (even or odd) becomes a polynomial of order n . The coefficient ν be given by $\nu = \nu_n$ with

$$\nu_n = \kappa[n(n+1) + g_\kappa(2n+1)] \quad (n \text{ is an integer number}).$$

The polynomial solutions are given by

- Even index (even power polynomials): The expressions of the first solutions $\mathcal{P}_j(y)$, in the particular cases of $j = 0, 2, 4$, are given by:

$$\begin{aligned} \mathcal{P}_0 &= 1, \\ \mathcal{P}_2 &= 1 - \kappa(3 + 2g_\kappa)y^2, \\ \mathcal{P}_4 &= 1 - 2\kappa(5 + 2g_\kappa)y^2 + \left(\frac{\kappa^2}{3}\right)(5 + 2g_\kappa)(7 + 2g_\kappa)y^4 \end{aligned} \quad (14)$$

- Odd index (odd power polynomials): The expressions of the second solutions $\mathcal{P}_j(y)$, for $j = 1, 3, 5$, are given by:

$$\begin{aligned} \mathcal{P}_1 &= y, \\ \mathcal{P}_3 &= y - \left(\frac{\kappa}{3}\right)(5 + 2g_\kappa)y^3, \\ \mathcal{P}_5 &= y - \left(\frac{2\kappa}{3}\right)(7 + 2g_\kappa)y^3 + \left(\frac{\kappa^2}{15}\right)(7 + 2g_\kappa)(9 + 2g_\kappa)y^5. \end{aligned} \quad (15)$$

5 Wavefunctions and eigenvalues

Let us start pointing out that the measure $d\mu_\kappa$ can be written as follows

$$d\mu_\kappa = \left(\frac{1}{\sqrt{1 - \kappa r^2}}\right) dx dy = \left(\frac{dz_x}{\sqrt{1 - \kappa z_x^2}}\right) dy. \quad (16)$$

Thus, the coordinates (z_x, y) also factorize the κ -dependent measure.

5.1 Sturm-Liouville problem for the Z -equation

The κ -dependent differential equation

$$a_0 Z'' + a_1 Z' + a_2 Z = 0,$$

with

$$a_0 = 1 - \kappa z_x^2, \quad a_1 = -\kappa z, \quad a_2 = \mu,$$

is not self-adjoint since $a'_0 \neq a_1$ but it can be reduced to self-adjoint form by making use of the following integrating factor

$$\mu(z_x) = \left(\frac{1}{a_0}\right) e^{\int (a_1/a_0) dz_x} = \sqrt{1 - \kappa z_x^2},$$

in such a way that if we denote by $q = q(z_x, \kappa)$ and $r = r(z_x, \kappa)$ the following functions

$$\begin{aligned} q(z_x, \kappa) &= e^{\int (a_1/a_0) dy} = \sqrt{1 - \kappa z_x^2}, \\ r(z_x, \kappa) &= \left(\frac{a_2}{a_0}\right) e^{\int (a_1/a_0) dx} = \frac{\mu}{\sqrt{1 - \kappa z_x^2}}. \end{aligned}$$

then we arrive to the following expression

$$\frac{d}{dz_x} \left[q(z_x, \kappa) \frac{dZ}{dz} \right] + r(z_x, \kappa) Z = 0. \quad (17)$$

Thus, this self-adjoint equation together with appropriate conditions for the behaviour of the solutions at the end points, constitute a Sturm-Liouville problem.

If κ is positive the range of the variable z_x is limited by the restriction $z_x^2 < 1/\kappa$. In this case the problem, defined in the bounded interval $[-a_\kappa, a_\kappa]$ with $a_\kappa = 1/\sqrt{\kappa}$, is singular because the function $q(z_x, \kappa)$ vanishes in the two end points $z_{x1} = -a_\kappa$ and $z_{x2} = a_\kappa$. So the first condition to be imposed is that the solutions $Z(z_x, \kappa)$ of the problem must be bounded functions at the two end points of the interval so that the norm be finite. Then we note that this situation is rather similar to the case of a particle in a one-dimensional square well with perfectly rigid impenetrable walls at the points $z_{x1} = -a_\kappa$ and $z_{x2} = a_\kappa$. Hence, taking into account that $u(z_{1x}) = -g_\kappa \frac{\pi}{2}$ and $u(z_{2x}) = g_\kappa \frac{\pi}{2}$, the application of the boundary conditions at $z_{x1,2} = \pm a_\kappa$, gives

$$\tilde{A} \cos\left(g_\kappa \frac{\pi}{2}\right) + \tilde{B} \sin\left(g_\kappa \frac{\pi}{2}\right) = 0, \quad \tilde{A} \cos\left(g_\kappa \frac{\pi}{2}\right) - \tilde{B} \sin\left(g_\kappa \frac{\pi}{2}\right) = 0,$$

from which we obtain two possibilities

$$\begin{aligned} \tilde{B} = 0 \quad \text{and} \quad \cos\left(g_\kappa \frac{\pi}{2}\right) &= 0, \\ \tilde{A} = 0 \quad \text{and} \quad \sin\left(g_\kappa \frac{\pi}{2}\right) &= 0. \end{aligned}$$

There are therefore two possible classes of solutions

(a) The coefficient g_κ and quantum number μ are given by

$$g_\kappa = g_{\kappa a} = 2m + 1, \quad \mu = \mu_a = \kappa (2m + 1)^2$$

(b) The coefficient g_κ and quantum number μ are given by

$$g_\kappa = g_{\kappa b} = 2m, \quad \mu = \mu_b = \kappa (2m)^2$$

In the case (a) the wave functions are given by

$$Z_{ma}(z_x) = \tilde{A} \cos\left((2m+1) \arcsin(\sqrt{\kappa} z_x)\right).$$

In the case (b) we obtain

$$Z_{mb}(z_x) = \tilde{B} \sin\left((2m) \arcsin(\sqrt{\kappa} z_x)\right).$$

Proposition 1 *The eigenfunctions $Z_{ma}(z_x)$ and $Z_{mb}(z_x)$ of the problem (17) are orthogonal in the interval $[-a_\kappa, a_\kappa]$ with respect to the function $r = \mu/(1 - \kappa z_x^2)$.*

Proof: This statement is just a consequence of the properties of the Sturm-Liouville problems.

5.2 Sturm-Liouville problem for the Y -equation

The κ -dependent differential equation

$$a_0 p'' + a_1 p' + a_2 p = 0,$$

with

$$a_0 = 1 - \kappa y^2, \quad a_1 = -2\kappa(1 + g_\kappa)y, \quad a_2 = \nu - \kappa g_\kappa,$$

is not self-adjoint since $a'_0 \neq a_1$ but it can be reduced to self-adjoint form by making use of an appropriate integrating factor in such a way that we arrive to

$$\frac{d}{dy} \left[q(y, \kappa) \frac{dp}{dy} \right] + r(y, \kappa) p = 0. \quad (18)$$

with $q = q(y, \kappa)$ and $r = r(y, \kappa)$ given by

$$q(y, \kappa) = (1 - \kappa y^2)^{1+g_\kappa}, \quad r(y, \kappa) = (\nu - \kappa g_\kappa)(1 - \kappa y^2)^{g_\kappa},$$

Thus, this self-adjoint equation together with appropriate conditions for the behaviour of the solutions at the end points, constitute a Sturm-Liouville problem.

If κ is positive the range of the variable y is limited by the restriction $y^2 < 1/\kappa$. In this case the problem, defined in the bounded interval $[-a_\kappa, a_\kappa]$ with $a_\kappa = 1/\sqrt{\kappa}$, is singular because the function $q(y, \kappa)$ vanishes in the two end points $y_1 = -a_\kappa$ and $y_2 = a_\kappa$.

From a purely mathematical viewpoint the eigenfunctions must be finite when $y \rightarrow \pm 1/\sqrt{\kappa}$ (a continuous function in a closed interval is always bounded and integrable). In addition, from a quantum viewpoint, the wave functions $Y(y)$ must vanish when $y \rightarrow \pm 1/\sqrt{\kappa}$. But this second stronger condition is satisfied because of the factor $(1 - \kappa y^2)^{(1/2)g_\kappa}$ in the expression of $Y(y)$.

The eigenvalues are the quantized values of the parameter ν , that is,

$$\nu_{na} = \kappa[n(n+1) + g_{\kappa a}(2n+1)], \quad g_{\kappa a} = 2m+1, \quad n = 0, 1, 2, \dots$$

$$\nu_{nb} = \kappa[n(n+1) + g_{\kappa b}(2n+1)], \quad g_{\kappa b} = 2m, \quad n = 0, 1, 2, \dots$$

and the eigenfunctions the associated polynomial solutions.

Proposition 2 *The eigenfunctions of the problem (18) are orthogonal in the interval $[-a_\kappa, a_\kappa]$ with respect to the function $r = (1 - \kappa y^2)^{g_\kappa}$.*

Proof: This statement is just a consequence of the properties of the Sturm-Liouville problems.

Because of this the polynomial solutions \mathcal{P}_{mn} , $n = 0, 1, 2, \dots$ of the equation (18) satisfy

$$\int_{-a_\kappa}^{a_\kappa} \mathcal{P}_{mn_1}(y, \kappa) \mathcal{P}_{mn_2}(y, \kappa) (1 - \kappa y^2)^m dy = 0, \quad n_1 \neq n_2, \quad \kappa > 0.$$

The first even $\mathcal{P}_{mj}(y)$, $j = 0, 2, 4$, and odd $\mathcal{P}_{mj}(y)$, $j = 1, 3, 5$, polynomials of this orthogonal family are

1. Even index (even power polynomials)

$$\begin{aligned} \mathcal{P}_{m0} &= 1, \\ \mathcal{P}_{m2} &= 1 - \kappa(3 + 2m)y^2, \\ \mathcal{P}_{m4} &= 1 - 2\kappa(5 + 2m)y^2 + \left(\frac{\kappa^2}{3}\right)(5 + 2m)(7 + 2m)y^4 \end{aligned}$$

2. Odd index (odd power polynomials)

$$\begin{aligned} \mathcal{P}_{m1} &= y, \\ \mathcal{P}_{m3} &= y - \left(\frac{\kappa}{3}\right)(5 + 2m)y^3, \\ \mathcal{P}_{m5} &= y - \left(\frac{2\kappa}{3}\right)(7 + 2m)y^3 + \left(\frac{\kappa^2}{15}\right)(7 + 2m)(9 + 2m)y^5. \end{aligned}$$

If we define the κ -dependent functions \mathcal{Y}_{mn} by

$$\mathcal{Y}_{mn}(y, \kappa) = \mathcal{P}_{mn}(y, \kappa) (1 - \kappa y^2)^{m/2}, \quad n = 0, 1, 2, \dots$$

then the above statement admits the following alternative form: *The κ -dependent functions $\mathcal{Y}_{mn}(y, \kappa) = \mathcal{Y}_n(y, \kappa, m)$, $n = 0, 1, 2, \dots$ are orthogonal with respect to the weight function $\tilde{r} = 1$:*

$$\int_{-a_\kappa}^{a_\kappa} \mathcal{Y}_{mn_1}(y, \kappa) \mathcal{Y}_{mn_2}(y, \kappa) dy = 0, \quad n_1 \neq n_2, \quad \kappa > 0,$$

(the same m in the two factors). Note that the orthogonality of the functions $\mathcal{Y}_{mn}(y, \kappa)$ coincides with the orthogonality with respect to the y -dependent second factor of the measure $d\mu_\kappa$ discussed in the first paragraph of Sec. 5.

5.3 Final solution

The wave functions of the κ -dependent free particle in the sphere S_κ^2 , when written as functions of (z_x, y) , $z_x = x/\sqrt{1 - \kappa y^2}$, corresponding to the first form $\Psi(e_1, e_{2j})$, are given (up to a multiplicative constant) by

$$\Psi_{mnr}(z_x, y) = Z_{mr}(z_x) Y_{mn}(y, \kappa), \quad r = a, b, \quad m, n = 0, 1, 2, \dots$$

with

$$\begin{aligned} Z_{ma}(z_x) &= \cos\left((2m+1) \arcsin(\sqrt{\kappa} z_x)\right), \quad m = 0, 1, 2, \dots \\ Z_{mb}(z_x) &= \sin\left((2m) \arcsin(\sqrt{\kappa} z_x)\right), \quad m = 0, 1, 2, \dots \end{aligned}$$

and

$$Y_{mn}(y, \kappa) = \mathcal{P}_{mn}(y, \kappa)(1 - \kappa y^2)^{m/2}, \quad m, n = 0, 1, 2, \dots$$

with energies given by

$$\begin{aligned} e_{m,n,a} = \mu_{ma} + \nu_{mna} &= \kappa(2m+1)^2 + \kappa[n(n+1) + (2m+1)(2n+1)] \\ &= \kappa(2m+n+1)(2m+n+2) = \kappa(N+1)(N+2) \\ e_{m,n,b} = \mu_{mb} + \nu_{mnb} &= \kappa(2m)^2 + \kappa[n(n+1) + (2m)(2n+1)] \\ &= \kappa(2m+n)(2m+n+1) = \kappa N(N+1) \end{aligned}$$

So the total energy

$$E_{m,n,r} = \left(\frac{\hbar^2}{2m}\right) e_{m,n,r}, \quad r = a, b, \quad m, n = 0, 1, 2, \dots$$

is proportional to the curvature κ and depends only of total quantum number N given by $N = 2m + n$.

To sum up, the quantum free particle on the sphere S_κ^2 is endowed with an infinite sequence of discrete energy values that can be considered as a consequence of the compact nature of the space. The energy levels are not equally spaced and the gap ΔE , between two consecutives levels, is proportional to N . We recall that an energy eigenvalue E is said to be degenerate when two or more independent eigenfunctions correspond to it. We have obtained that the values of $E_{m,n}$ depend only on N so they are degenerate with respect m and n . Next we present the wavefunctions corresponding to the three lowest values of the energy:

- (i) The fundamental level, with energy given by $e = 2\kappa$, is non-degenerate and represented by only one wavefunction

$$\Psi_{00a}(z_x, y) = Z_{0a}(z_x) Y_{00}(y, \kappa) = \cos(\arcsin(\sqrt{\kappa} z_x)).$$

- (ii) The second level, with energy given by $e = 6\kappa$, is represented by the following two wavefunctions

$$\begin{aligned} \Psi_{01a}(z_x, y) &= Z_{0a}(z_x) Y_{01}(y, \kappa) = \cos(\arcsin(\sqrt{\kappa} z_x)) y. \\ \Psi_{10b}(z_x, y) &= Z_{1b}(z_x) Y_{10}(y, \kappa) = \sin(2 \arcsin(\sqrt{\kappa} z_x)) (1 - \kappa y^2)^{1/2}. \end{aligned}$$

- (iii) The third level, with energy given by $e = 12\kappa$, is represented by the following three wavefunctions

$$\begin{aligned} \Psi_{02a}(z_x, y) &= Z_{0a}(z_x) Y_{02}(y, \kappa) = \cos(\arcsin(\sqrt{\kappa} z_x)) (1 - 3\kappa y^2). \\ \Psi_{10a}(z_x, y) &= Z_{1a}(z_x) Y_{10}(y, \kappa) = \cos(3 \arcsin(\sqrt{\kappa} z_x)) (1 - \kappa y^2)^{1/2}. \\ \Psi_{11b}(z_x, y) &= Z_{1b}(z_x) Y_{11}(y, \kappa) = \sin(2 \arcsin(\sqrt{\kappa} z_x)) y (1 - \kappa y^2)^{1/2}. \end{aligned}$$

When we consider the Euclidean limit then, as $\kappa \rightarrow 0$, all these normalizable wave functions $\Psi_{m,n}$ disappear and the associated energies $E_{m,n}$ vanish. This means that the Euclidean wave planes characterized by a continuous spectrum cannot be obtained as the limit of the discrete normalizable $\kappa > 0$ spectrum. This situation can be considered as a consequence of the boundary conditions; that is, the $\kappa \rightarrow 0$ limit of the general solutions of the κ -dependent equations (without introducing boundary conditions in the points $-a_\kappa$ and a_κ) are the Euclidean wave planes but once the boundary conditions are introduced in the points $-a_\kappa$ and a_κ then the result is a discrete $\kappa > 0$ spectrum that cannot be related with the $\kappa = 0$ description.

Finally, we mention that another approach for the free motion in the sphere S^3 , in terms of spectrum generating algebras, has been recently developed in [74].

6 Hyperbolic $\kappa < 0$ case

6.1 Resolution of the Z -equation

If we assume a negative value for the curvature then we have $\kappa = -|\kappa| < 0$ and the equation for Z becomes

$$(1 + |\kappa| z_x^2) Z'' + (|\kappa| z_x) Z' + \mu Z = 0. \quad (19)$$

Then assuming for Z an expression of the form

$$Z = e^{i u(z_x)},$$

with $u(z_x)$ a function to be determined, we obtain that the general solution of (19) is given by

$$Z = A e^{i u(z_x)} + B e^{-i u(z_x)}, \quad u = \frac{\sqrt{\mu}}{\sqrt{|\kappa|}} \operatorname{arcsinh}(\sqrt{|\kappa|} z_x), \quad \kappa = -|\kappa| < 0, \quad (20)$$

that is a well defined function for all the values of z_x (we recall that in this case there are not restrictions for the domain of z_x). This solution satisfies the appropriate Euclidean limit

$$\lim_{\kappa \rightarrow 0} Z = A e^{i k_x x} + B e^{-i k_x x}, \quad k_x = \sqrt{\mu},$$

and therefore it can be considered as representing a κ -dependent hyperbolic deformation of the Euclidean plane-wave solution.

6.2 Resolution of the Y -equation

The equation for the function Y takes now the form

$$(1 + |\kappa| y^2) Y'' + (2|\kappa| y) Y' + \mu |\kappa| \left(\frac{y^2}{1 + |\kappa| y^2} \right) Y + \nu Y = 0, \quad \kappa = -|\kappa| < 0. \quad (21)$$

In order to obtain an hypergeometric equation, similar to the spherical Eq. (13), we can consider the following factorization

$$Y(y) = p(y) (1 + |\kappa| y^2)^{(1/2)g},$$

but now we arrive to the condition $g^2|\kappa| + \mu = 0$ which leads to

$$g = i g_\kappa, \quad g_\kappa = \sqrt{\frac{\mu}{|\kappa|}}.$$

The consequence is that the new equation must be complex. In fact if we assume the complex factorization

$$Y(y) = \mathcal{P}(y) (1 + |\kappa| y^2)^{(i/2)g_\kappa}, \quad \mathcal{P} = p_1(y) + i p_2(y),$$

then we obtain

$$(1 + |\kappa| y^2) \mathcal{P}_{yy} + 2|\kappa| (1 + i g_\kappa) y \mathcal{P}_y + (\nu + i |\kappa| g_\kappa) \mathcal{P} = 0, \quad (22)$$

that represents a complex hypergeometric equation. Alternatively it can be written as a system of two coupled real equations.

$$\begin{aligned} (1 + |\kappa| y^2) p''_{1yy} + 2|\kappa| y p'_{1y} + \nu p_1 &= |\kappa| g_\kappa (2y p'_{2y} + p_2) \\ (1 + |\kappa| y^2) p''_{2yy} + 2|\kappa| y p'_{2y} + \nu p_2 &= -|\kappa| g_\kappa (2y p'_{1y} + p_1) \end{aligned}$$

So in this case there are two possibilities: (i) to solve directly the Eq. (21) (power series solution) or (ii) to solve the complex hypergeometric equation (22) (both equations satisfy correctly the Euclidean limit). In any case it can be proved the nonexistence of polynomial solutions for real values of the quantum number ν . This means that the eigenvalues ν can take any positive value and the spectrum for the energy is continuous as in the Euclidean case.

We note that in this hypergeometric case the two variables, z_x and y , are defined in the whole real line and both functions, $Z(z_x)$ and $Y(y)$, turn out to be nonnormalizable functions. So, in a sense, this hyperbolic case can be considered as more similar to the Euclidean one than the spherical $\kappa > 0$ one.

Finally, let us mention the study by Balazs and Voros [55] of quantum mechanics on the hyperbolic plane. It is concerned, for the most part, with the study of chaos in compact manifolds with constant negative curvature which arise as quotients of the hyperbolic plane (called pseudosphere in [55]) by suitable discrete groups of isometries. In particular, they discuss a pseudosphere analogous of the standard Euclidean plane waves, precisely those eigenfunctions of the corresponding Laplace-Beltrami operator which separate in horospherical coordinates. These can be imagined as the (suitably rescaled) limits of pseudospherical circular waves when the sink or source point goes to infinity. The solutions of the Schrodinger equation we are discussing here are neither circular waves nor horospherical waves, because these allow separation of variables in a variant of parallel coordinates. As mentioned in the introduction, and remarked also in [55] the idea of ‘plane waves in a manifold of constant curvature’ admits several possible realizations in spaces of constant negative curvature, and while the horospherical waves are somehow more natural than others (because there is still a source or sink at some point, albeit at infinity), the solutions we are dealing with here are a different possibility and would have a

family of geodesics orthogonal to given geodesic as wavefronts. In the $\kappa \rightarrow 0$ limit, these (as well as horospherical plane waves) can be expected to collapse to Euclidean plane waves. A more detailed study of solutions of the curved Laplace-Beltrami equations in different coordinates will be done elsewhere.

7 κ -dependent Schrödinger equation II

The second alternative way of solving the quantum κ -dependent problem is to consider the system of the two following Schrödinger equations

$$(\widehat{H}_1 + \kappa \widehat{J}^2) \Psi = e_{1j} \Psi, \quad \widehat{H}_2 \Psi = e_2 \Psi,$$

that can be solved by using the property of separability of the equation $\widehat{H} \Psi = E \Psi$ in coordinates (x, z_y) with z_y defined as $z_y = y/\sqrt{1 - \kappa x^2}$.

Thus, if we assume that $\Psi(x, z_y)$ is a function of the form

$$\Psi(x, z_y) = X(x)Z(z_y),$$

then we arrive to

$$(1 - \kappa z_y^2) Z'' - (\kappa z_y) Z' + \mu' Z = 0,$$

and

$$(1 - \kappa x^2) X'' - (2\kappa x) X' - \mu' \kappa \left(\frac{x^2}{1 - \kappa x^2} \right) X + \nu' X = 0.$$

where the two eigenvalues μ' and ν' are related with the two partial energies e_2 and e_{1j} by

$$\mu' = \left(\frac{2m}{\hbar^2} \right) e_2, \quad \nu' = \left(\frac{2m}{\hbar^2} \right) e_{1j}.$$

These two equations can be solved by repeating the previous analysis with the appropriate interchange of variables. We only recall that this second approach leads to a value of E given by $E = e_{1j} + e_2$ and that the solution $\Psi(x, z_y)$ can also be denoted by $\Psi(e_{1j}, e_2)$.

8 Final comments and outlook

Let us summarize our results. We have studied the quantum free particle on spherical and hyperbolic spaces using a curvature dependent approach. In the first part of the paper, that was mainly concerned with geometrical questions, an important point was the identification of the three Killing vectors and the associated Noether symmetries. This was important for the quantization procedure that was carried out in two steps:

- (i) Quantization of the three Noether momenta as self-adjoint operators with respect to a κ -dependent measure (of course, when $\kappa = 0$ we recover the standard quantization of the linear and the angular momenta).

- (ii) Construction of the quantum Hamiltonian $\widehat{H}(\kappa)$ as a function of the three operators \widehat{P}_1 , \widehat{P}_2 , and \widehat{J} .

The second part of the paper was devoted to the resolution of the κ -dependent equations. The separation of the Schrödinger equation in coordinates (z_x, y) introduces the term depending of the angular momentum \widehat{J} , that plays the role of an effective potential, in the the y -equation in such a way that

- (i) The motion along the z_x direction is a (κ -dependent) free motion and the solution is a κ -deformed plane wave.
- (ii) The motion along the y direction leads to an hypergeometric equation.

What introduce differences between the $\kappa > 0$ and the $\kappa < 0$ cases is that in the spherical case the space is compact and this leads to a Sturm-Liouville problem with boundary conditions rather similar to to the case of a quantum particle in a one-dimensional square well with perfectly rigid impenetrable walls. The result is a discrete spectrum with normalizable wave functions $\Psi_{m,n}$ and associated energies $E_{m,n}$ when $\kappa > 0$.

We finalize pointing out two questions to be studied. First, as was stated in Sec. (2) this problem can also be solved by using κ -dependent spherical coordinates that corresponds to the approach $(\widehat{H}_1 + \widehat{H}_2) \Psi = e_{12} \Psi$, $\widehat{J}^2 \Psi = e_j \Psi$ (the solutions must be κ -deformations of the standard Euclidean spherical waves). Second, we have obtained, as a mathematical by-product of this formalism, a κ -dependent family of orthogonal polynomials. They deserve a deeper mathematical study.

9 Appendix. Geodesic parallel coordinates

Suppose M be a 2-dimensional Riemannian manifold, O a point on M and g_1 and g_2 , two orthogonal geodesics through O . Let P be an arbitrary point, in some suitable neighbourhood of O , and denote by P_1 and P_2 the orthogonal projections of P on g_1 and g_2 (that is, P_1 is the intersection of g_1 with the geodesic through P orthogonal to g_1). Then we can characterize the point P by

1. The two distances (u, y_κ) defined as follows: u is the distance of O to P_1 (measured along g_1) and y_κ the distance of P_1 to P (measured along the geodesic by P and P_1).
2. The two distances (x_κ, v) defined as follows: x_κ is the distance of P_2 to P (measured along the geodesic by P and P_2) and v the distance of O to P_2 (measured along g_2).

In the first case we have the parallel coordinates of P relative to (O, g_1) and in the second case relative to (O, g_2) [75]. In the (u, y_κ) system the curves ' $u = \text{constant}$ ' are geodesics and the curves ' $y_\kappa = \text{constant}$ ' meet these geodesics orthogonally. In the (x_κ, v) system the geodesics are

the curves ‘ $v = \text{constant}$ ’ and ‘ $x_\kappa = \text{constant}$ ’. Notice that in the general case we have $u \neq x_\kappa$ and $v \neq y_\kappa$.

In the case of M being a space of constant curvature κ , the (u, y_κ) and (x_κ, v) expressions for the differential arc length element ds_κ^2 are given by

$$ds_\kappa^2 = C_\kappa^2(y_\kappa) du^2 + dy_\kappa^2, \quad \text{and} \quad ds_\kappa^2 = dx_\kappa^2 + C_\kappa^2(x_\kappa) dv^2, \quad (23)$$

so that in both cases we get $ds^2 = ds_0^2 = dx^2 + dy^2$ for the particular value $\kappa = 0$ characterizing the Euclidean case. These two systems, although different for $\kappa \neq 0$, can be related by using formulae of spherical and hyperbolic trigonometry for $\kappa > 0$ and for $\kappa < 0$ respectively.

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References

- [1] N.D. Birrell and P.C. Davies, *Quantum fields in curved space* (Cambridge Univ. Press, new edition, 1994).
- [2] L. E. Parker and D. J. Toms, *Quantum field theory in curved spacetime* (Cambridge Univ. Press, 2009).
- [3] A. Comtet, Ann. Physics **173**, 185–209 (1987).
- [4] G.V. Dunne, Ann. Physics **215**, 233–263 (1992).
- [5] S. Kim and Ch. Lee, Ann. Physics **296**, 390–405 (2002).
- [6] H. Fakhri and M. Shariati, J. Phys. A **37**, L539–L545 (2004).
- [7] V.V. Gritsev and Yu.A. Kurochkin, Phys. Rev. B **64**, 035308 (2001).
- [8] A.A. Bogush, V.V. Gritsev, Yu.A. Kurochkin, and V.S. Otchik, Phys. Atomic Nuclei **65**, 1052–1056 (2002).
- [9] D.V. Bulaev, V.A. Geyler, and V.A. Margulis, Phys. Rev. B **69**, 195313 (2004).
- [10] C. Furtado, A. Rosas, and S. Azevedo, EPL **79**, 57001 (2007).
- [11] P. Stovicek and M. Tusek, in *Recent Advances in Operator Theory in Hilbert and Krein Spaces*, 291–304, Oper. Theory Adv. Appl. vol. 198, Birkhauser-Verlag, Bassel 2010. arXiv:0811.3825

- [12] E. Schrödinger, Proc. Roy. Irish Acad. Sect. A. **46**, 9–16 (1940).
- [13] L. Infeld, Phys. Rev. **59**, 737–747 (1941).
- [14] A.F. Stevenson, Phys. Rev. **59**, 842–843 (1941).
- [15] L. Infeld and A. Schild, Phys. Rev. **67**, 121–122 (1945).
- [16] A.O. Barut, A. Inomata, and G. Junker, J. Phys. A **20**, 6271–6280 (1987).
- [17] A.O. Barut, A. Inomata, and G. Junker, J. Phys. A **23**, 6271–6280 (1990).
- [18] L.M. Nieto, M. Santander, and H.C. Rosu, Mod. Phys. Lett. A **14**, 2463–2469 (1999).
- [19] S. Bellucci and A. Nersessian, Phys. Rev. D **67**, 065013 (2003).
- [20] S. Bellucci, A. Nersessian, and A. Yeranyan, Phys. Rev. D **70**, 085013 (2004).
- [21] L. Mardoyan and A. Nersessian, Phys. Rev. B **72**, 233303 (2005).
- [22] P.R. Giri, J. Phys. A **40**, 3539–3547 (2007).
- [23] C. Grosche, Fortschritte der Physik **42**, 509–584 (1994).
- [24] A. Lecheheb, M. Merad, and T. Boudjedaa, Ann. Physics **322**, 1233–1246 (2007).
- [25] P.W. Higgs, J. Phys. A **12**, 309–323 (1979).
- [26] H.I. Leemon, J. Phys. A **12**, 489–501 (1979).
- [27] P. Leal Ferreira, J. Phys. A **16**, 2087–2091 (1983).
- [28] A.O. Barut and R. Wilson, Phys. Lett. A **110**, 351–354 (1985).
- [29] N. Katayama, Nuovo Cimento B **107**, 763–768 (1992).
- [30] D. Bonatsos, C. Daskaloyannis, and K. Kokkotas, Phys. Rev. A **48**, R3407–R3410 (1993).
- [31] D. Bonatsos, C. Daskaloyannis, and K. Kokkotas, Phys. Rev. A **50**, 3700–3709 (1994).
- [32] E.G. Kalnins, W. Miller, and G. S. Pogosyan, J. Math. Phys. **41**, 2629–2657 (2000).
- [33] J.J. Slawianowski, Rep. Math. Phys. **46**, 429–460 (2000).
- [34] V.V. Gritsev, Yu.A. Kurochkin, and V.S. Otchik, J. Phys. A **33**, 4903–4910 (2000).
- [35] A. Nersessian and G. Pogosyan, Phys. Rev. A **63**, 020103 (2001).
- [36] M.F. Rañada and M. Santander, J. Math. Phys. **43**, 431–451 (2002).
- [37] M.F. Rañada and M. Santander, J. Math. Phys. **44**, 2149–2167 (2003).

- [38] T.G. Vozmischeva, *Integrable problems of celestial mechanics in spaces of constant curvature*, Astrophysics and Space Science Library vol. 295 (Kluwer Academic Pub., Dordrecht, 2003).
- [39] D. Voyatzi and S. Ichtiaroglou, *Celestial Mech. Dynam. Astronom.* **93**, 331–342 (2005).
- [40] A.V. Borisov and I.S. Mamaev, *Celestial Mech. Dynam. Astronom.* **96**, 1–17 (2006).
- [41] J.F. Cariñena, M.F. Rañada, and M. Santander, *J. Phys. A* **40**, 13645–13666 (2007).
- [42] P. Stovicek and M. Tusek, *Russian J. of Math. Phys.* **14**, 493–497 (2007).
- [43] J.F. Cariñena, M.F. Rañada, and M. Santander, *J. Math. Phys.* **49**, 032703 (2008).
- [44] M. Gadella, J. Negro, G.P. Pronko, and M. Santander, *J. Phys. A* **41**, 304030 (2008).
- [45] V.V. Gritsev and Yu.A. Kurochkin, *J. Phys. A* **33**, 4073–4079 (2000).
- [46] V.V. Kudryashov, Yu.A. Kurochkin, E.M. Ovsyuk, and V.M. Red'kov, *SIGMA* **6**, 004, 34 pp. (2010).
- [47] C. Grosche, G.S. Pogosyan, and A.N. Sissakian, *Fortschr. Phys.* **43**, 523–563 (1995).
- [48] M.F. Rañada and M. Santander, *J. Math. Phys.* **40**, 5026–5057 (1999).
- [49] E.G. Kalnins, J.M. Kress, G. Pogosyan, and W. Miller, *J. Phys. A* **34**, 4705–4720 (2001).
- [50] E.G. Kalnins, J.M. Kress, and P. Winternitz, *J. Math. Phys.* **43**, 970–983 (2002).
- [51] A. Ballesteros, F.J. Herranz, M. Santander, and T. Sanz-Gil, *J. Phys. A* **36**, L93–L99 (2003).
- [52] F.J. Herranz and A. Ballesteros, *SIGMA Symmetry Integrability Geom. Methods Appl.* **2**, 010 (2006).
- [53] A. Ballesteros, A. Enciso, F.J. Herranz, and O. Ragnisco, *J. Nonlinear Math. Phys.* **15**, supplement 3, 43–52 (2008).
- [54] A. Ballesteros and F.J. Herranz, *J. Phys. A* **42**, 245203 (2009).
- [55] N.L. Balazs and A. Voros, *Phys. Reports* **143**, 109–240 (1986).
- [56] M.E. Spina and M. Saraceno, *J. Phys. A* **32**, 7803–7813 (1999).
- [57] J.F. Cariñena, M.F. Rañada, M. Santander, and M. Senthilvelan, *Nonlinearity* **17**, 1941–1963 (2004).
- [58] P. Dombrowski and J. Zitterbarth, *Demonstratio Mathematica* **24**, 375–458 (1991).
- [59] J.F. Cariñena, M.F. Rañada, and M. Santander, *Ann. of Physics* **322**, 434–459 (2007).
- [60] J.F. Cariñena, M.F. Rañada, and M. Santander, *Ann. of Physics* **322**, 2249–2278 (2007).

- [61] J.F. Cariñena, M.F. Rañada, and M. Santander, J. Math. Phys. **48**, 102106 (2007).
- [62] P.M. Mathews and M. Lakshmanan, Quart. Appl. Math. **32**, 215–218 (1974).
- [63] M. Lakshmanan and S. Rajasekar, *Nonlinear dynamics. Integrability, chaos and patterns*, Advanced Texts in Physics (Springer-Verlag, Berlin, 2003).
- [64] P. Létourneau and L. Vinet, Ann. Phys. **243**, 144–168 (1995).
- [65] J. Hietarinta, Phys. Lett. A **246**, 97–104 (1998).
- [66] M.B. Sheftel, P. Tempesta, and P. Winternitz, J. Math. Phys. **42**, 659–673 (2001).
- [67] C. Daskaloyannis, J. Math. Phys. **42**, 1100–1119 (2001).
- [68] S. Gravel and P. Winternitz, J. Math. Phys. **43**, 5902–5912 (2002).
- [69] M.A. Rodriguez and P. Winternitz, J. Math. Phys. **43**, 1309–1322 (2002).
- [70] J. Bérubé and P. Winternitz, J. Math. Phys. **45**, 1959–1973 (2004).
- [71] A. Ballesteros, F. Herranz, and O. Ragnisco, J. Phys. A **38**, 7129–7144 (2005).
- [72] E.G. Kalnins, W. Miller, and G.S. Pogosyan, J. Math. Phys. **47**, 033502 1–30 (2006).
- [73] J.A. Calzada, J. Negro, and M.A. del Olmo, J. Math. Phys. **47**, 043511 1–17 (2006).
- [74] M. Gadella, J. Negro, L.M. Nieto, G.P. Pronko, and M. Santander, J. Math. Phys. (to appear, 2011). arXiv:1007.1423
- [75] W. Klinberger, *A course in differential geometry*, Graduate texts in Mathematics (Springer-Verlag, New York, 1978).